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ORIGINAL PAPER

The homotopy analysis method applied to the Kolmogorov–Petrovskii–Piskunov (KPP) and fractional KPP equations

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Abstract In this paper, the Homotopy analysis method (HAM) is employed to obtain the analytical/numerical solutions for linear and nonlinear Kolmogorov Petrovskii-Piskunov (KPP) and fractional KPP equations. The proposed method is a powerful and easy-to-use analytical tool for linear and nonlinear problems. This method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. Some illustrative examples are presented. Moreover the use of HAM is found to be accurate, simple, convenient, flexible and computationally attractive.

Keywords KPP equation · Fractional KPP equation · Homotopy analysis method

1 Introduction

The linear and nonlinear reaction-diffusion equations play fundamental role in a great number of various models of reaction-diffusion processes, mathematical biology, chemistry, and genetics and so on. Thus, one of these diffusion equations is the Kolmogorov–Petrovskii–Piskunov (KPP) equation. It has special importance in science and engineering and constitutes an excellent model for many systems in various fields. The subject of fractional calculus and its applications (that is, the theory of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, mainly due to its applications in diverse fields of science and engineering. Recently, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives [7]. There has been

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some excellent attempt to solve linear problems with multiple fractional derivatives (the so-called multi-term equations) [7,4]. Approximate and analytical methods have included the Adomian decomposition method (ADM) [3], the Homotopy perturbation method (HPM) [5], the Variational iteration method (VIM) [18], and the Homotopy analysis method (HAM) [13].

The Homotopy analysis method (HAM) [13,14,12,10,11,8] is a general analytic scheme to get series solutions of various types of linear and nonlinear equations. In this work, we will implement the Homotopy analysis method (HAM) to obtain the numerical solutions of the following linear and linear KPP and fractional KPP equations of the form considered in [16]. As to the KPP equations we refer to the papers [2,15,9].

(i) The linear KPP equation:

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}+f(U) \tag{1}$$

(ii) The multiple-term fractional KPP equation:

$$\frac{\partial^\alpha U}{\partial t^\alpha}=\frac{\partial^2 U}{\partial x^2}-2U^3, \quad t>0, 0<\alpha\leq 1. \tag{2}$$

(iii) The nonlinear KPP equation with time and space fractional derivatives:

$$\frac{\partial^\alpha U}{\partial t^\alpha}=\frac{\partial^{2\beta} U}{\partial x^{2\beta}}-2U^3, \quad t>0, \alpha>0, \beta\leq 1. \tag{3}$$

2 Definitions of fractional derivatives and integrals

In this section, we present some notations, definitions and preliminary facts that will be used further in this work. Fractional calculus is 300 years old topic. The first serious attempt to give logical definition is due to Liouville. Since then several definitions of fractional integrals and derivatives have been proposed. These definitions include the Riemann-Liouville, the Caputo, the Weyl, the Hadamard, the Marchaud, the Riesz, the Grunwald-Letnikov and Erdelyi-Kober. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physically interpretations. Therefore, in this paper we shall use the Caputo derivative D^α proposed by Caputo in his work on the theory of viscoelasticity.

In the development of theories of fractional derivatives and integrals, it appears many definitions such as Riemann-Liouvlle and Caputo fractional differential-integral definition as follows.

(1) Riemann-Liouville definition:

$${}_a^R D_t^\alpha f(t)=\begin{cases}\frac{d^m f(t)}{dt^m}, & \alpha=m\in N;\\ \frac{d^m}{dt^m}\frac{1}{\Gamma(m-\alpha)}\int\limits_a^t \frac{f(T)}{(t-T)^{\alpha-m+1}}dT, & 0\leq m-1<\alpha<m.\end{cases}$$

Fractional integral of order α is as follows:

$$ {}_{a}^{R}I_{t}^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int\limits_{0}^{t} (t-T)^{-\alpha-1} f(T)dT, \quad \alpha < 0. $$

(2) Caputo definition:

$$ {}_{a}^{c}D_{t}^{\alpha} f(t) = \begin{cases} \frac{d^m f(t)}{dt^m}, & \alpha = m \in N; \\ \frac{1}{\Gamma(m-\alpha)} \int\limits_{a}^{t} \frac{f^{(m)}(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \le m-1 < \alpha < m. \end{cases} $$

3 Basic idea of homotopy analysis method (HAM)

In this section the basic ideas of the homotopy analysis method are introduced. Here a description of the method is given to handle the general nonlinear problem.

$$ Nu_0(t) = 0, t > 0 \tag{4} $$

where $\boldsymbol{N}$ is a nonlinear operator and $\boldsymbol{u_0(t)}$ is unknown function of the independent variable t.

3.1 Zero-order deformation equation

Let $\boldsymbol{u_0(t)}$ denote the initial guess of the exact solution of Eq. (1), h $\neq \mathbf{0}$ an auxiliary parameter, $\boldsymbol{H(t)} \neq \mathbf{0}$ an auxiliary function and L is an auxiliary linear operator with the property.

$$ \boldsymbol{L(f(t))} = \mathbf{0}, \quad \boldsymbol{f(t)} = \mathbf{0}. \tag{5} $$

The auxiliary parameter h, the auxiliary function $\boldsymbol{H(t)}$, and the auxiliary linear operator $\boldsymbol{L}$ play an important role within the HAM to adjust and control the convergence region of solution series. Liao [13,14,12] constructs, using $\boldsymbol{q} \in [\mathbf{0}, \mathbf{1}]$ as an embedding parameter, the so-called zero-order deformation equation.

$$ \boldsymbol{(1\text{-}q)L}[(\emptyset\boldsymbol{(t;q)} - \boldsymbol{u_0}(t)] = \boldsymbol{qhH(t)N}[(\emptyset\boldsymbol{(t;q)}], \tag{6} $$

where $\emptyset\boldsymbol{(t;q)}$ is the solution which depends on h, $\boldsymbol{H(t),L}$, $\boldsymbol{u_0(t)}$ and q. When q = 0, the zero-order deformation Eq. (6) becomes

$$ \emptyset(\boldsymbol{t}; \mathbf{0}) = \boldsymbol{u_0(t)}, \tag{7} $$

And when q = 1, since h $\neq \mathbf{0}$ and $\boldsymbol{H(t)} \neq \mathbf{0}$, the zero-order deformation Eq. (1) reduces to,

$$\boldsymbol{N}[\emptyset(\boldsymbol{t;1})] = \mathbf{0}, \tag{8}$$

So, $\emptyset(\boldsymbol{t;1})$ is exactly the solution of the nonlinear equation. Define the so-called mth order deformation derivatives.

$$\boldsymbol{u_m(t)} = \frac{\mathbf{1}}{\boldsymbol{m!}}\frac{\partial^{\boldsymbol{m}}\emptyset\boldsymbol{(t;q)}}{\partial \boldsymbol{q^m}} \tag{9}$$

If the power series Eq. (9) of $\emptyset\boldsymbol{(t;q)}$ converges at q = 1, then we gets the following series solution:

$$\boldsymbol{u(t)} = \boldsymbol{u_0(t)} + \sum_{\boldsymbol{m=1}}^{\infty}\boldsymbol{u_m(t)}. \tag{10}$$

where the terms $\boldsymbol{u_m(t)}$ can be determined by the so-called high order deformation described below.

3.2 High- order deformation equation

Define the vector,

$$\overrightarrow{\boldsymbol{u_n}}\boldsymbol{=}\{\boldsymbol{u_0(t)}, \boldsymbol{u}_1\boldsymbol{(t)}, \boldsymbol{u_2}(\boldsymbol{t})\ldots\ldots\boldsymbol{u_n(t)} \tag{11}$$

Differentiating Eq. (6) m times with respect to embedding parameter q, the setting q = 0 and dividing them by $\boldsymbol{m}!$, we have the so-called mth order deformation equation.

$$\boldsymbol{L}\left[\boldsymbol{u_m(t)} - \aleph_{\boldsymbol{m}}\boldsymbol{u_{m-1}}(\boldsymbol{t})\right] = \boldsymbol{hH(t)R_m}(\overrightarrow{\boldsymbol{u_m}}, \boldsymbol{t}), \tag{12}$$

where

$$\aleph_{\boldsymbol{m}} = \begin{cases} \boldsymbol{o},\ \boldsymbol{m} \le \mathbf{1} \\ \mathbf{1},\ \boldsymbol{otherwise} \end{cases} \tag{13}$$

and

$$\boldsymbol{R_m}(\overrightarrow{\boldsymbol{u_m}}, \boldsymbol{t}) = \frac{1}{(\boldsymbol{m}-\mathbf{1})!}\frac{\partial^{\boldsymbol{m-1}}\boldsymbol{N}[\emptyset\boldsymbol{(t;q)}]}{\partial\boldsymbol{q^{m-1}}} \tag{14}$$

For any given nonlinear operator $\boldsymbol{N}$, the term $\boldsymbol{R_m}(\overrightarrow{\boldsymbol{u_m}}, \boldsymbol{t})$ can be easily expressed by (14). Thus, we can gain $\boldsymbol{u_1(t)}, \boldsymbol{u}_2(\boldsymbol{t})\ldots\ldots$.. by means of solving the linear high-order deformation with one after the other order in order. The mth –order approximation of u (t) is given by

$$u(t) = \sum_{k=0}^{m} u_k(t) \tag{15}$$

ADM, VIM and HPM are special cases of HAM when we set $h = -1$ and $H(r, t) = 1$. We will get the same solutions for all the problems by above methods when we set h $= -1$ and $H(r, t) = 1$. When the base functions are introduced the $H(r, t) = 1$ is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence.

4 Solving linear KPP equation by the Homotopy analysis method (HAM)

Consider the linear KPP equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u, \quad x, t \in R \tag{16}$$

Subject to the initial condition

$$u(x, 0) = e^{-x} + x, x \in R \tag{17}$$

We apply the Homotopy analysis method to Eqs. (16) and (17), as follows: since $m \geq 1$, $\chi_m = 1$and set h $= -1$ and H(r,t) $= 1$ in Eq. (12), then it becomes

$$u_m(x, t) = u_{m-1}(x, 0) - L^{-1}(\Re_m(u_{m-1}, x, t)) \tag{18}$$

where

$$\Re_m(u_{m-1}, x, t) = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2} + u_{m-1} \tag{19}$$

and the initial condition

$$u_0(x, t) = e^{-x} + x \tag{20}$$

We can obtain the results using Eq. (19)

$$u_1(x, t) = e^{-x} + x - xt \tag{21}$$

$$u_2(x, t) = e^{-x} + x - xt + \frac{xt^2}{2} \tag{22}$$

$$u_3(x, t) = e^{-x} + x - xt + \frac{xt^2}{2} - \frac{1}{6xt^3} \tag{23}$$

$$u_4(x, t) = e^{-x} + x - xt + \frac{xt^2}{2} - \frac{1}{6xt^3} + \frac{1}{24xt^4} \tag{24}$$

Then the final solution is

$$u(x, t) = e^{-x} + x\left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \dots\right)$$
$$u(x, t) = e^{-x} + xe^{-t}. \tag{25}$$

Example 4.1 We consider the following reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2tu, \quad x, t \in R \tag{26}$$

Subject to the initial condition

$$u(x, 0) = e^x, x \in R \tag{27}$$

We apply the Homotopy analysis method to Eqs. (26) and (27) as follows:
since $m \geq 1$, $\chi_m = 1$ and set h $= -1$ and H(r,t) $= 1$ in Eq. (12), then it becomes

$$u_m(x, t) = u_{m-1}(x, 0) - L^{-1}(\Re_m(u_{m-1}, x, t)) \tag{28}$$

where

$$\Re_m(u_{m-1}, x, t) = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial^2 u_{m-1}}{\partial x^2} + 2tu_{m-1} \tag{29}$$

and the initial condition

$$u_0(x, t) = e^x \tag{30}$$

We can obtain the following results using Eq. (29)

$$u_1(x, t) = e^x(1 + t + t^2) \tag{31}$$

$$u_2(x, t) = e^x\left(1 + t + \frac{t^2}{2!}\right)(1 + t^2) \tag{32}$$

$$u_3(x, t) = e^x\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}\right)\left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!}\right) \tag{33}$$

Then the final solution in a closed form is

$$u(x, t) = e^{x+t+t^2}. \tag{34}$$

Homotopy analysis method (HAM) provides to adjust and control the convergence rate of the solution in the particular region with h.

5 The HAM for the multiple-term fractional KPP equation

In this section, to establish the effectiveness and the applicability of our approach, we will implement the HAM to construct numerical solutions for the multiple-term fractional KPP equation in the form [16]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} - 2u^3, t > 0, 0 < \alpha \leq 1, \tag{35}$$

Subject to the initial condition

$$u(x,0) = \frac{2xcs\left(x^2, \frac{1}{\sqrt{2}}\right)}{dn\left(x^2, \frac{1}{\sqrt{2}}\right)}, \tag{36}$$

where $cs\left(x^2, \frac{1}{\sqrt{2}}\right)$ and $dn\left(x^2, \frac{1}{\sqrt{2}}\right)$ are Jacobi elliptic functions.

To solve the above problem by HAM, we select the auxiliary parameters as follows:

$$L_F(\phi(x,t;p)) = D_t^{\alpha}\left[\phi\left(x,t;p\right)\right] \tag{37}$$

with the property $L_F(c_1) = 0$.

Using the above definition, we construct the zeroth-order deformation equations

$$(1-p)L_F\left[\phi(x,t;p) - \chi_m u_0(x,t)\right] = phN_F\left[\phi(x,t;p)\right] \tag{38}$$

Obviously, when $p = 0$ and $p = 1$.

$$\phi(x,t;0) = u_0(x,t), \phi(x,t;1) = u(x,t) \tag{39}$$

Differentiating the zeroth-order deformation Eq. (38) m times with respect to p, and finally dividing by m!, we have

$$L_F(u_m(x,t) - \chi_m u_{m-1}(x,t)) = h\Re_m\left[\overrightarrow{u_{m-1}}\right] \tag{40}$$

where

$$\Re_m\left[\overrightarrow{u_{m-1}}\right] = D_t^{\alpha}u_{m-1}\left(x,t\right) + \frac{\partial^2 u_{m-1}}{\partial x^2} - 2u_{m-1}^3 \tag{41}$$

On applying the operator J_t^{α} both sides of the Eq. (40), we get

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hJ_t^{\alpha}\Re\left[\overrightarrow{u_{m-1}}\right] \tag{42}$$

Subsequently solving mth-order deformation equations one has

$$u_0(x,t) = \frac{2xcs\left(x^2, \frac{1}{\sqrt{2}}\right)}{dn\left(x^2, \frac{1}{\sqrt{2}}\right)}, \tag{43}$$

$$u_1(x,t) = \frac{6xt^{\alpha}\left[2 - 2sn^2\left(x^2, \frac{1}{\sqrt{2}}\right) + sn^4\left(x^2, \frac{1}{\sqrt{2}}\right)\right]}{sn^2\left(x^2, \frac{1}{\sqrt{2}}\right)dn^2\left(x^2, \frac{1}{\sqrt{2}}\right)\Gamma\left(\alpha+1\right)}, \tag{44}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and so on. Then $u(x,t) = u_o + u_1 + u_2 + \ldots$

6 The HAM for the time and space fractional nonlinear KPP equations

We consider the time and space fractional nonlinear KPP equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + 2u^3, t > 0, 0 < \alpha, \beta \leq 1. \tag{45}$$

With the initial condition

$$u(x, 0) = x^2. \tag{46}$$

To solve the above problem by HAM, we select the auxiliary parameters as follows:

$$L_F(\phi(x, t; p)) = D_t^{\alpha} [\phi(x, t; p)] \tag{47}$$

with the property $L_F(c_1) = 0$.

Using the above definition, we construct the zeroth-order deformation equations

$$(1 - p)L_F [\phi (x, t; p) - \chi_m u_0 (x, t)] = phN_F [\phi(x, t; p)] \tag{48}$$

Obviously, when $p = 0$ and $p = 1$.

$$\phi(x, t; 0) = u_0(x, t), \phi(x, t; 1) = u(x, t) \tag{49}$$

Differentiating the zeroth-order deformation Eq. (38) m times with respect to p, and finally dividing by m!, we have

$$L_F(u_m(x, t) - \chi_m u_{m-1}(x, t)) = h\Re_m \left[\overrightarrow{u_{m-1}}\right] \tag{50}$$

where

$$\Re_m \left[\overrightarrow{u_{m-1}}\right] = D_t^{\alpha} u_{m-1} (x, t) - \frac{\partial^{2\beta} u_{m-1}}{\partial x^{2\beta}} - 2u_{m-1}^3 \tag{51}$$

On applying the operator J_t^{α} both sides of the Eq. (40), we get

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + hJ_t^{\alpha}\Re \left[\overrightarrow{u_{m-1}}\right] \tag{52}$$

Subsequently solving mth-order deformation equations one has

$$u_0(x, t) = x^2. \tag{53}$$

$$u_1(x, t) = \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \left[\frac{2x^{2-2\beta}}{\Gamma(3 - 2\beta)} - 2x^6\right] \tag{54}$$

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and so on. Then we can calculate the approximate solution of the above problem. Our results can be compared with K. A. Gepreel results [5].

7 Conclusion

In this work, the HAM is used to obtain the approximate/analytical solutions of the various linear and nonlinear Kolmogorov–Petrovskii–Piskunov (KPP) and fractional KPP equations with initial conditions. This scheme provides us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. In conclusion, HAM gives accurate approximate solution for nonlinear problems in comparison with other methods.

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